PRACTICAL ESTIMATES OF THE ERRORS ASSOCIATED WITH THE GOVERNING SHEAROGRAPHY EQUATION

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INTRODUCTION

In a series of papers Hung[1-3] pioneered the development of shearography, an optical NDE technique that detects gradients of surface displacements. Its utility for qualitative flaw characterization has been demonstrated, and while there is a need for using shearography in NDE for quantitative analysis, a large amount of the research[2-7] has concentrated on the qualitative evaluation of structures and materials. The purpose of this paper is to begin building upon a foundation for the newly emerging quantitative shearography[8].

We begin the analysis by considering the approximations leading to the governing shearography equation,

$$\Delta\phi = \frac{2\pi}{\lambda}\left(A\frac{\partial u}{\partial x} + B\frac{\partial v}{\partial x} + C\frac{\partial w}{\partial x}\right)\Delta x, \tag{1}$$

where $\Delta\phi$ is the relative phase change due to displacements between two nearby points on the surface of a deformed test sample, λ is the wavelength of the coherent beam of light, u, v, and w are the surface displacements in the x,y and z directions respectively, Δx is the camera shear and the coefficients A,B and C are geometric functions of x, y and z.

We then identify and quantify the error terms that arise from this approximate relation. Due to the complexity of the equations a rigorous analysis does not seem possible. We instead settle for a formal analysis which we feel provides valuable information concerning the limitations of shearography. We show that for typical problems (displacements or displacement gradients << camera shearing) the predominant error term in this approximation is relatively small provided the curvature of the displacement field is not that large. However, if one further simplifies the equation by considering only the out-of-plane displacements (as is often done in many commercial instruments) the error can become significant. This error depends upon geometric and physical parameters and, as we show, can be estimated with the aid of a simple formula.

DERIVATION OF THE GOVERNING EQUATION

For clarity we shall briefly repeat part of the derivation of (1) using an approach slightly different from Hung's. The approach was chosen to make the analysis of the error terms as convenient as possible. For a more complete derivation see Hung[3].

Shearography is based on the phenomenon that coherent waves of light having different path lengths interfere and produce a fringe pattern. This fringe pattern represents a change in phase and can be shown to be proportional to the first partial derivatives of the deformed objects surface displacements.

Fig. 1 shows two points $P\,(x, y, z)$ and $Q\,(x + \Delta x, y, z)$ on the surface of an object which, upon deformation, move to the points $P'\,(x + u, y + v, z + w)$ and $Q'\,(x + \Delta x + u\,(x + \Delta x, y, z), y + v\,(x + \Delta x, y, z), z + w\,(x + \Delta x, y, z))$ respectively. The change in the path length of a ray of light coming from a source S(x,y,z) and reflected from the point P(x,y,z), on the surface, to the observer at O(x,y,x), due to the deformation, is given by

$$\Delta l_1 = [(SP' + P'O) - (SP + PO)]. \tag{2}$$

Similarly, for the sheared point Q(x+Δx,y,z) the change in path length is

$$\Delta l_2 = [(SQ' + Q'O) - (SQ + QO)]. \tag{3}$$

In shearography, the interference of the sheared and unsheared waves gives rise to a phase change given by,

$$\Delta\phi = \frac{2\pi}{\lambda}(\Delta l_2 - \Delta l_1). \tag{4}$$

To derive (1) and examine the errors inherent in its derivation it is convenient to define the following auxiliary function,

$$\begin{aligned} F(\Delta x) &\equiv \Delta l_2 - \Delta l_1 \\ &= [(x - x_0 + \Delta x + \bar{u})^2 + (y - y_0 + \bar{v})^2 + (z - z_0 + \bar{w})^2]^{1/2} \\ &\quad + [(x - x_s + \Delta x + \bar{u})^2 + (y - y_s + \bar{v})^2 + (z - z_s + \bar{w})^2]^{1/2} \end{aligned}$$

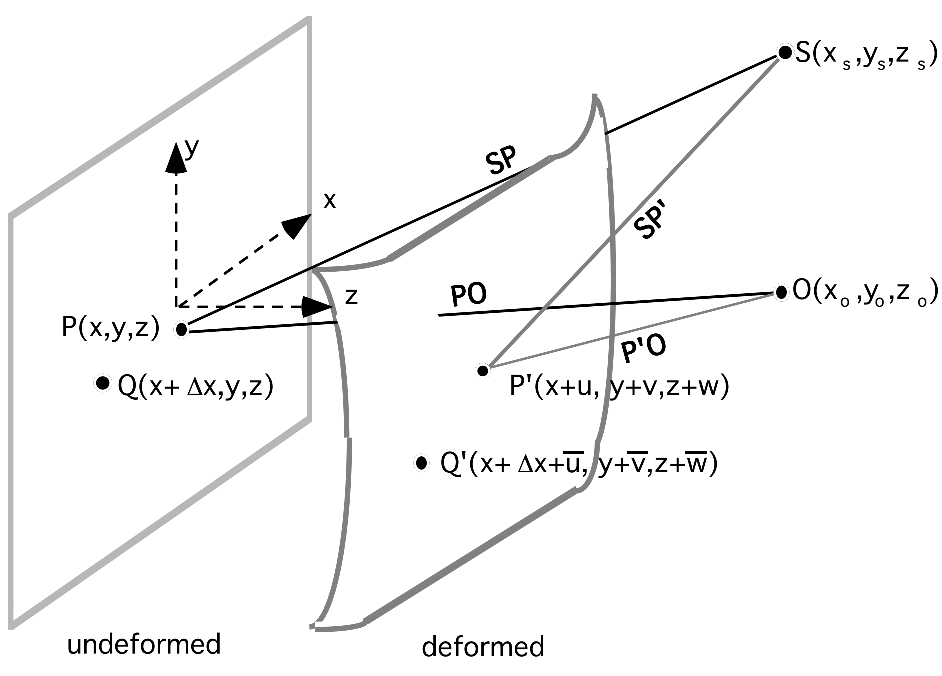

Fig. 1. Optical path differences between deformed and undeformed configurations.

$$
\begin{aligned}
&-[\,(x-x_0+\Delta x)^2+(y-y_0)^2+(z-z_0)^2\,]^{1/2}\\
&-[\,(x-x_S+\Delta x)^2+(y-y_S)^2+(z-z_S)^2\,]^{1/2}\\
&-\{\,[\,(x-x_0+u)^2+(y-y_0+v)^2+(z-z_0+w)^2\,]^{1/2}\\
&\quad+[\,(x-x_s+u)^2+(y-y_s+v)^2+(z-z_s+w)^2\,]^{1/2}\\
&\quad-[\,(x-x_0)^2+(y-y_0)^2+(z-z_0)^2\,]^{1/2}\\
&\quad-[\,(x-x_S)^2+(y-y_S)^2+(z-z_S)^2\,]^{1/2}\,\}
\end{aligned}
\tag{5}
$$

where, $\bar{u} \equiv u(x+\Delta x, y, z)$, $\bar{v} \equiv v(x+\Delta x, y, z)$, $\bar{w} \equiv w(x+\Delta x, y, z)$, $u = u(x, y, z)$, $v = v(x, y, z)$ and $w = w(x, y, z)$. The resulting expressions on the right hand side are obtained by substituting the appropriate distances from Fig. 1, into Δl_1 and Δl_2 above.

To help facilitate the error analysis, which will require the explicit computation of higher order terms, we take an approach different from Hung's[1] and expand (5) in a Taylor series in Δx about the point $P'(x+u(x,y,z), y+v(x,y,z), z+w(x,y,z))$ to obtain

$$
\begin{aligned}
F(\Delta x) &= F(0) + \frac{\partial F}{\partial(\Delta x)}(0)\,\Delta x + \frac{\partial^2 F}{\partial(\Delta x)^2}(0)\,\frac{(\Delta x)^2}{2!} + \ldots\\
&= \left(A\frac{\partial u}{\partial x} + B\frac{\partial v}{\partial x} + C\frac{\partial w}{\partial x}\right)\Delta x +\\
&\quad \left[A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 v}{\partial x^2} + C\frac{\partial^2 w}{\partial x^2} + 2\left(D\frac{\partial u}{\partial x} - G\frac{\partial v}{\partial x} - H\frac{\partial w}{\partial x}\right)\right]\frac{(\Delta x)^2}{2} +\\
&\quad \left[D\left(\frac{\partial u}{\partial x}\right)^2 + E\left(\frac{\partial v}{\partial x}\right)^2 + F\left(\frac{\partial w}{\partial x}\right)^2 - 2\left(G\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + H\frac{\partial u}{\partial x}\frac{\partial w}{\partial x} + I\frac{\partial v}{\partial x}\frac{\partial w}{\partial x}\right)\right]\frac{(\Delta x)^2}{2} + \ldots
\end{aligned}
\tag{6}
$$

where,

$$
\begin{aligned}
A &= \frac{(x-x_0+u)}{R_0} + \frac{(x-x_s+u)}{R_s}, \quad B = \frac{(y-y_0+v)}{R_0} + \frac{(y-y_s+v)}{R_s},\\
C &= \frac{(z-z_0+w)}{R_0} + \frac{(z-z_s+w)}{R_s},\\
D &= \frac{(y-y_0+v)^2+(z-z_0+w)^2}{R_0^3} + \frac{(y-y_s+v)^2+(z-z_s+w)^2}{R_s^3},\\
E &= \frac{(x-x_0+u)^2+(z-z_0+w)^2}{R_0^3} + \frac{(x-x_s+u)^2+(z-z_s+w)^2}{R_s^3},\\
F &= \frac{(y-y_0+v)^2+(x-x_0+u)^2}{R_0^3} + \frac{(y-y_s+v)^2+(x-x_s+u)^2}{R_s^3},\\
G &= \frac{(y-y_0+v)(x-x_0+u)}{R_0^3} + \frac{(y-y_s+v)(x-x_s+u)}{R_s^3},\\
H &= \frac{(x-x_0+u)(z-z_0+w)}{R_0^3} + \frac{(x-x_s+u)(z-z_s+w)}{R_s^3},\\
I &= \frac{(y-y_0+v)(z-z_0+w)}{R_0^3} + \frac{(y-y_s+v)(z-z_s+w)}{R_s^3},
\end{aligned}
\tag{7}
$$

$$R_0 = [(x - x_0 + u)^2 + (y - y_0 + v)^2 + (z - z_0 + w)^2]^{1/2}$$

and

$$R_s = [(x - x_s + u)^2 + (y - y_s + v)^2 + (z - z_s + w)^2]^{1/2}.$$

The term linear in Δx in (6) is proportional to the shearography equation. The next term in the expansion will be used to obtain an approximation for the error induced by replacing $(\Delta l_2 - \Delta l_1)$ with only the Δx term. The equation Hung derives is slightly different, but can be obtained from our results by expanding the coefficients A, B and C in terms of the displacements u, v and w. This would unnecessarily complicate our analysis, so it will not be done here.

A FORMAL ERROR ANALYSIS

1. The Error Associated with Truncating Terms of Order $(\Delta x)^2$ and Higher

The error is defined to be the difference between the exact change in path-length and the shearography approximation and is defined as

$$e = \left| \frac{\lambda}{2\pi} \left[(\Delta l_2 - \Delta l_1) - \left[A\frac{\partial u}{\partial x} + B\frac{\partial v}{\partial x} + C\frac{\partial w}{\partial x} \right] \Delta x \right] \right| \tag{8}$$

A complete analysis would require a rigorous bound on the higher order terms in the Taylor expansion, but due to the complexity of the equations, this does not seem feasible at the present time. However, it is useful to have some knowledge about the magnitude of the first order terms used in the shearography equation relative to the next order terms which are discarded. While a study of the relative size of the two terms will not enable us to fully quantify the error involved in the approximation, it will provide valuable information concerning the shearographic approximation.

With this in mind we consider the second order terms in (6) which we have separated into two parts, where the latter terms are all quadratic in the displacement gradients. For practical problems $\partial u/\partial x$, $\partial v/\partial x$, and $\partial w/\partial x << 1$, therefore, we shall ignore terms in (6) that are quadratic in displacement gradients compared to the linear terms. To analyze the resulting equation we consider two special cases in which we will first assume that the 2nd derivative, and then the 1st derivative terms dominate the $(\Delta x)^2$ coefficient.

Case 1

Assuming that the 2nd derivative terms dominate we ignore the gradients in the $(\Delta x)^2$ coefficient in (6). This situation could arise if the bending strain in the x direction, which is proportional to the curvature of the out-of-plane displacement field, is much larger than the in-plane strains.

Considering only the most dominant term from (6) we can approximate the error in (8) by

$$e \approx \bar{e} = \frac{\lambda}{2\pi} \left| \left(A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 v}{\partial x^2} + C\frac{\partial^2 w}{\partial x^2} \right) \frac{(\Delta x)^2}{2} \right| \tag{9}$$

To obtain a bound on this error let

$$M = \max\left(\left|\frac{\partial^2 u}{\partial x^2}\right|, \left|\frac{\partial^2 v}{\partial x^2}\right|, \left|\frac{\partial^2 w}{\partial x^2}\right| \right), \text{ over the region of interest,} \tag{10}$$

and substitute (10) into (9) to obtain,

$$\bar{e} \leq |A + B + C| \frac{M\lambda (\Delta x)^2}{4\pi}. \tag{11}$$

The expression within the absolute value symbol can be shown to be bounded, i.e., $-2 \le A + B + C \le 2$, and consequently, we obtain an approximate bound for the error

$$\bar{e} \le \frac{M\lambda (\Delta x)^2}{2\pi}. \tag{12}$$

If we now let

$$N = \max\left(\left|\frac{\partial u}{\partial x}\right|, \left|\frac{\partial v}{\partial x}\right|, \left|\frac{\partial w}{\partial x}\right| \right), \text{ over the region of interest,} \tag{13}$$

and consider the ratio of (12) with respect to the first two terms in (6) we have,

$$p_1 = \frac{\frac{M\lambda (\Delta x)^2}{2\pi}}{\frac{\lambda}{2\pi}[2N\Delta x + M(\Delta x)^2]} = \frac{M\Delta x}{2N + M\Delta x} = \frac{\Delta x}{2\left(\frac{N}{M}\right) + \Delta x}. \tag{14}$$

The above result is an approximation of the percentage of the optical signal due to the error term. If an estimate of the N/M term can be made, then a shear can be chosen to yield as small an error as desired. A plot of this function for various percentages of error, and different shears and N/M ratios is shown in Fig. 2. Even for relatively large shears ($\Delta x \ge 0.01$ meters the error does not become appreciable until M is about 20 times larger than N. Consequently, the approximation is a relatively good one provided the problem being studied does not contain bending strains that are significantly larger than the in-plane strains.

<u>Case 2</u>

Assuming that the gradients dominate we ignore the 2nd derivative terms in the $(\Delta x)^2$ coefficient in (6) and again take the ratio of the magnitude of the second term with respect to the magnitude of the entire expression to obtain:

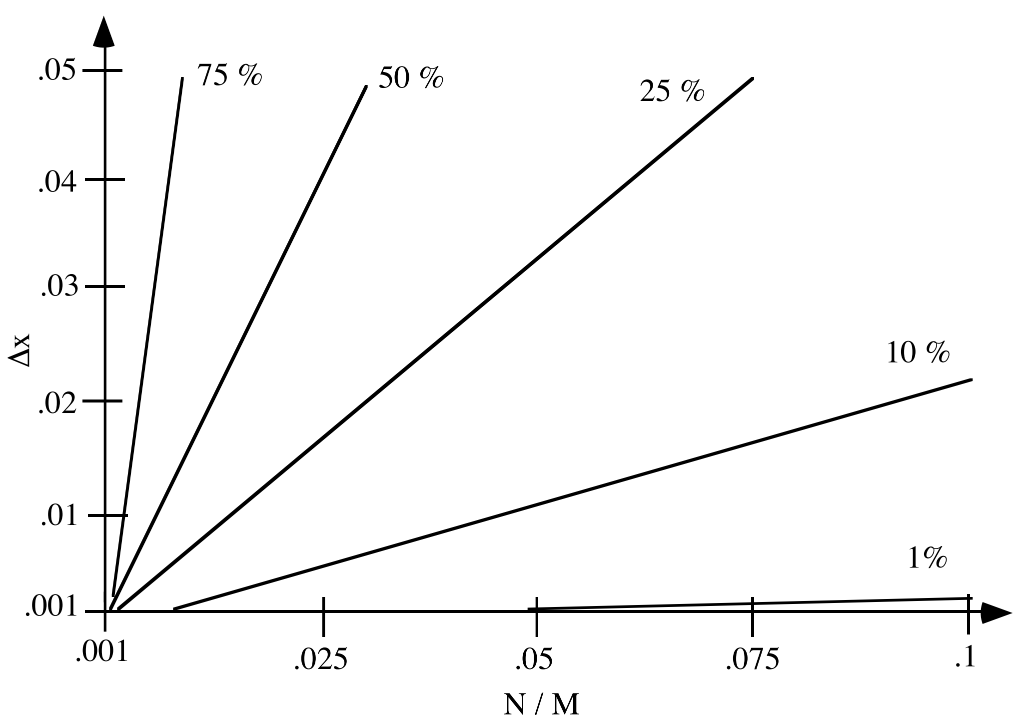

Fig. 2. Contour plot of (14) showing the relative error for different shears and N/M ratios.

$$
p_2 = \frac{\Delta x}{\left|\frac{A + B + C}{D - G - H}\right| + \Delta x} \tag{15}
$$

Considering the terms in (7) we see that the expression in absolute value in (15) is on the order of magnitude of the distance from the object or source to the sample. A contour plot of this relation for various relative errors is shown in Fig. 3. Thus, when the bending strains are not a factor in the problem we would either need to introduce a relatively large shear (> a few centimeters) or move the laser and camera extremely close (<.1 meters) to the test sample before a significant error would result.

<u>2. The Error Associated with Truncating the In-Plane Gradient Terms in the Shearography Equation</u>

Additional approximations are often employed to further simplify the shearography equation. For many practical problems the coefficients A and B are much smaller than C, and hence are often ignored.The question we now address is just how significant the error in making this assumption is. The relevant expression is given as

$$
\left(\frac{x - x_0 + u}{R_0} + \frac{x - x_s + u}{R_s}\right)\frac{\partial u}{\partial x} + \left(\frac{y - y_0 + v}{R_0} + \frac{y - y_s + v}{R_s}\right)\frac{\partial v}{\partial x} + \left(\frac{z - z_0 + w}{R_0} + \frac{z - z_s + w}{R_s}\right)\frac{\partial w}{\partial x} \tag{16}
$$

The displacements *u,v* and *w* are quite small in comparison to *x*, *y* and *z* for the field of interest and, consequently, can be ignored. To simplify the analysis further we will require that both the light source and camera be located at the center of a relatively flat test area a fixed distance $\bar{z}$ from the object, i.e. $x_0 = x_s = y_0 = y_s = z = 0$, $R_0 = R_s \equiv R$, and $z_0 = z_s \equiv \bar{z}$ (see Fig.4).

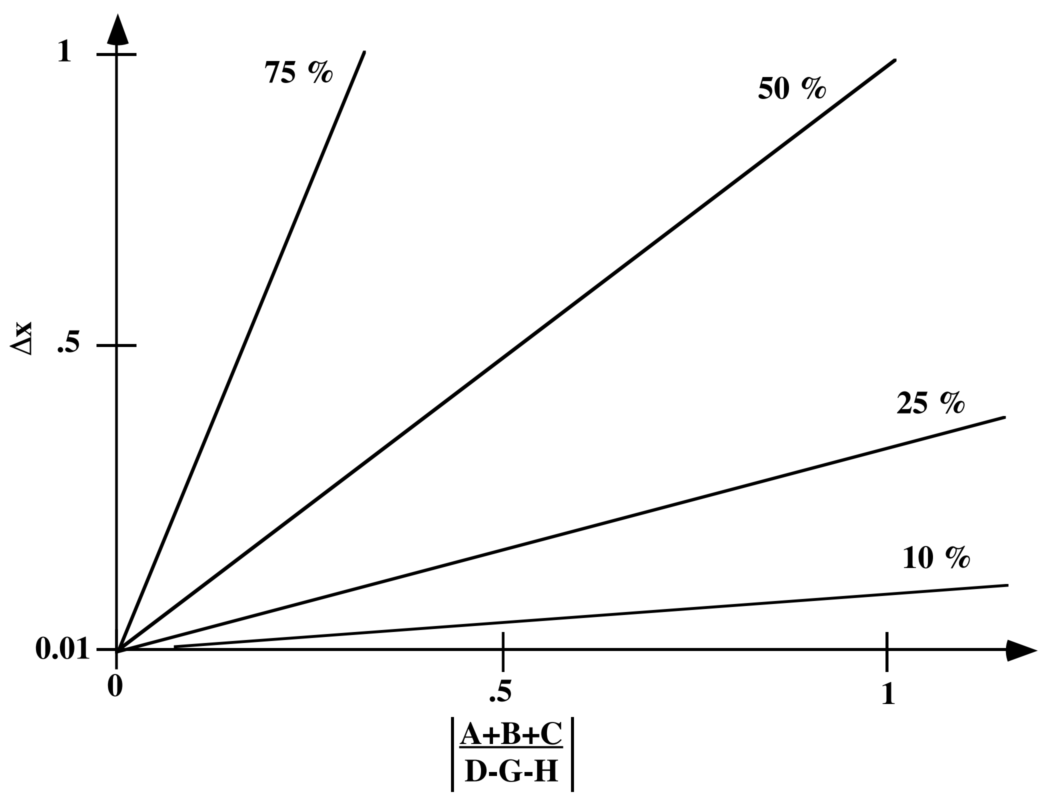

Fig. 3. Contour plot of (15) showing the relative error for different shears and distances from the test object.

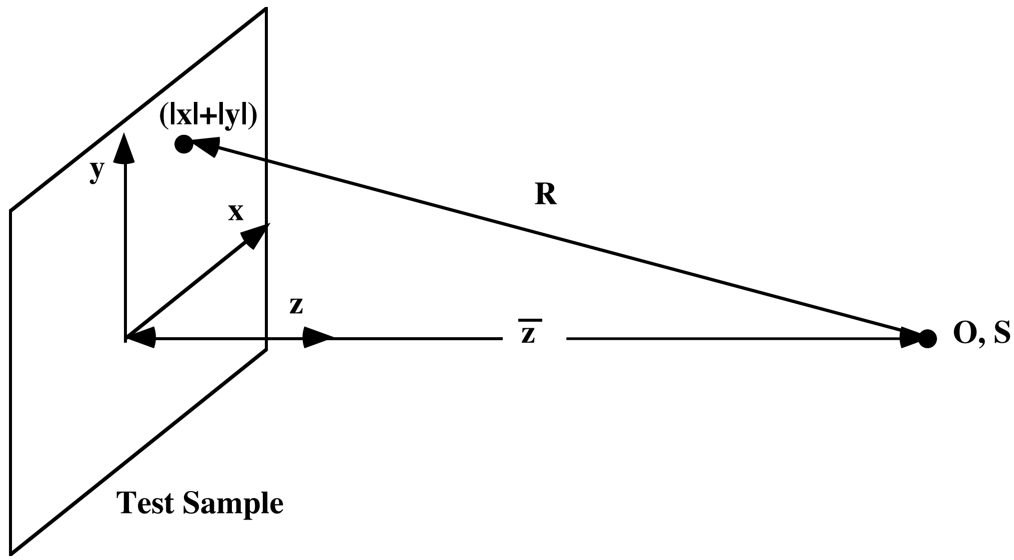

Fig. 4. Configuration for normal viewing and illumination

Using these values (16) then reduces to

$$\frac{2x}{\mathrm{R}}\frac{\partial u}{\partial x} + \frac{2y}{\mathrm{R}}\frac{\partial v}{\partial x} - \frac{2\bar{z}}{\mathrm{R}}\frac{\partial w}{\partial x} \tag{17}$$

Now letting

$$\mathrm{N} = \max\left(\left|\frac{\partial u}{\partial x}\right|, \left|\frac{\partial v}{\partial x}\right|\right), \text{ over the region of interest,} \tag{18}$$

and consider the ratio of the magnitude of the second order term with the magnitude of both terms to obtain

$$\mathrm{p}_3 = \frac{\frac{2\bar{z}}{R}\left|\frac{\partial w}{\partial x}\right|}{\frac{2N}{R}(|x| + |y|) + \frac{2\bar{z}}{R}\left|\frac{\partial w}{\partial x}\right|} = \frac{\left|\frac{\partial w}{\partial x}\right|/N}{\frac{(|x| + |y|)}{\bar{z}} + \left|\frac{\partial w}{\partial x}\right|/N}\,. \tag{19}$$

This represents the percentage of the expression due to the out-of-plane gradient term. Ideally, we would like this to be as close to unity as possible. A contour plot of this expression is shown in Fig. 5. The quantity (|x|+|y|)/$\bar{z}$, is the ratio of the distance from the center of the test sample with respect to the distance of the camera/laser to the test sample (see Fig. 4). As the field of view is increased the in-plane gradients can have a significant effect on the signal. This effect can become even more pronounced if their are relatively large in-plane strains. For example, if the camera/laser is about 1 meter from the test sample, and if the in-plane strains are about 20 times larger than the out-of-plane gradient, then at a radius of 50 mm from the center of the field of view only 50% of the signal is due to the out-of-plane gradient term. Thus, one needs to be aware that both in-plane and out-of-plane terms are present in the signal before a proper fringe analysis can be done.

SUMMARY

Using a non rigorous analysis we were able to show that the approximations leading to the governing shearography equation (1) introduce relatively small error terms for practical problems. In addition, this error is easily controlled by varying the shear. However, if the shearography equation is further simplified to approximate only out-of-plane gradients, then a significant error can result. As the field of view increases the in-plane gradients begin to effect and then eventually dominate the fringe pattern. A simple formula has been derived to obtain a rough estimate of the relative percentages of the signal due to in-plane and out-of-plane gradients. Thus, if a quantitative evaluation of the fringe pattern is desired it is necessary to either take these results into account or to alter the test to limit their effects.

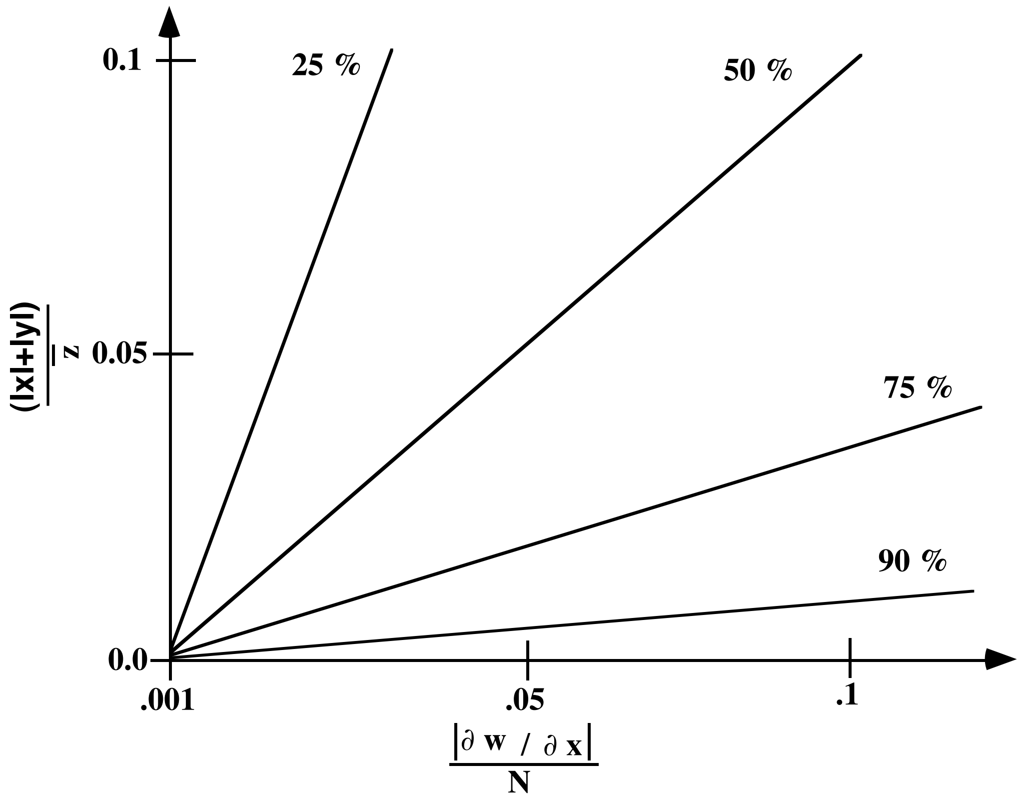

Fig. 5. Contour plot of (19) showing the percentage of the expression due to the out-of-plane gradient term as a function of the distance from the center of the test sample and the relative magnitude of the out-of-plane versus the in-plane gradients.

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